ON THE EXTRACTION OF ROOTS OF WHOLE NUMBERS BY THE "BINOMIAL THEOREM."

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In many instances the "Binomial Theorem" may be employed with advantage in the extraction of roots of whole numbers.

I.—We have

$$(a^n+x)^{\frac{1}{n}}=a+\frac{x}{n\,a^{n-1}}-\frac{(n-1)x^2}{1.\,2.\,n^2.\,a^{2n-1}}+\frac{(n-1)(2n-1)x^3}{1.\,2.\,3.\,n^3.\,a^{3n-1}}$$
$$-\frac{(n-1)(2n-1)(3n-1)x^4}{1.\,2.\,3.\,4.\,n^4.\,a^{4n-1}}+\text{\&c.}\ldots\ldots\ldots\ldots(1),$$

and

$$(a^n-x)^{\frac{1}{n}}=a-\frac{x}{n\,a^{n-1}}-\frac{(n-1)x^2}{1.\,2.\,n^2.\,a^{2n-1}}-\frac{(n-1)(2n-1)x^3}{1.\,2.\,3.\,n^3.\,a^{3n-1}}$$
$$-\frac{(n-1)(2n-1)(3n-1)\,x^4}{1.\,2.\,3.\,4.\,n^4.\,a^{4n-1}}-\text{\&c.}\ldots\ldots\ldots\ldots(2).$$

These formulas are very convenient when x is small compared with a.

EXAMPLES.—1. Required the square root of 26. Making $a=5$, $x=1$ and $n=2$, in (1),

$$\sqrt{26}=5+\frac{1}{10}-\frac{1}{1000}+\frac{1}{50000}-\frac{1}{2000000}+\text{\&c.},$$
$$=5.099019+.$$

2.—Required the square root of 216.

$$216=24\times 9;\quad\therefore\sqrt{216}=3\sqrt{24}.$$

By (2),

$$\sqrt{24}=5-\frac{1}{10}-\frac{1}{1000}-\frac{1}{50000}-\frac{1}{2000000}-\text{\&c.},$$
$$=4.8989794+;$$
$$\therefore\sqrt{216}=14.696938+.$$

3.—Required the fifth root of 30.

$$\sqrt[5]{30}=\sqrt[5]{32-2}=2-\frac{1}{5.2^3}-\frac{1}{5^2.2^6}-\frac{3}{5^3.2^{10}}-\text{\&c.},$$
$$=1.974351+.$$

Also,

$$(a^n + x)^{-\frac{1}{n}} = \frac{1}{a} - \frac{x}{n\,a^{n+1}} + \frac{(n+1)\,x^2}{1.\;2.\;n^2.\;a^{2n+1}} - \frac{(n+1)\,(2n+1)x^3}{1.\;2.\;3.\;n^3.\;a^{3n+1}}$$

$$+ \frac{(n+1)\;(2\;n+1)\,(3\;n+1)x^4}{1.\;2.\;3.\;4.\;n^4.\;a^{4n+1}} - \&\text{c}\ldots\ldots(3),$$

and

$$(a^n - x)^{-\frac{1}{n}} = \frac{1}{a} + \frac{x}{n\,a^{n+1}} + \frac{(n+1)\,x^2}{1.\;2.\;n^2.\;a^{2n+1}} + \frac{(n+1)\;(2\;n+1)x^3}{1.\;2.\;3.\;n^3.\;a^{3n+1}}$$

$$+ \frac{(n+1)\,(2\;n+1)\;(3\;n+1)x^4}{1.\;2.\;3.\;4.\;n^4.\;a^{4n+1}} + \&\text{c}\ldots\ldots(4).$$

Example.—Required the one-hundredth root of 2.

$$2 = \frac{1}{1-\frac{1}{2}} = (1-\tfrac{1}{2})^{-1}, \;\therefore (2)^{\frac{1}{100}} = (1-\tfrac{1}{2})^{-\frac{1}{100}}.$$

Therefore by (4)

$$(2)^{\frac{1}{100}} = 1 + \frac{1}{200} + \frac{101}{80000} + \frac{6767}{16000000} + \frac{2036867}{12800000000} + \&\text{c}.,$$

$$= 1.00695 +, \text{ using 9 terms.}$$

II.—Let a be any number, then

$$a = \frac{a\,p^n}{p^n} \times \frac{q^n}{q^n} = \frac{p^n}{q^n}\left[1 - \left(\frac{p^n - a\,q^n}{p^n}\right)\right];$$

$$\therefore (a)^{\frac{1}{n}} = \frac{p}{q}\left[1 - \left(\frac{p^n - a\,q^n}{p^n}\right)\right]^{\frac{1}{n}},$$

$$= \frac{p}{q}\left[1 - \left(\frac{p^n - a\,q^n}{n p^n}\right) - \frac{(n-1)}{1.\;2}\left(\frac{p^n - a\,q^n}{n\,p^n}\right)^2\right.$$

$$\left. - \frac{(n-1)\,(2\,n-1)}{1.\;2.\;3}\left(\frac{p^n - a\,q^n}{n\,p^n}\right)^3 - \frac{(n-1)(2n-1)(3n-1)}{1.\;2.\;3.\;4}\left(\frac{p^n - a\,q^n}{n\,p^n}\right)^4 - \&\text{c}.\right],$$

where p and q may be any numbers chosen at pleasure.

Let r_m be the root of a to m places of decimals, and R_m the remainder; then if $p=(10)^m$ and

$$q = \frac{(10)^m\;r_m}{a},$$

$$(a)^{\frac{1}{n}} = \frac{(10)^m}{(10)^m\left(\frac{r_m}{a}\right)}\left[1 - \frac{\left(\frac{R_m}{a}\right)}{(10)^{2m}}\right]^{\frac{1}{n}}$$

Take $a = 2$, $n = 2$ and $m = 8$; then

$$\sqrt{2} = \frac{100000000}{70710678}\left[1 - \frac{33560632}{10000000000000000}\right]^{\frac{1}{2}},$$

$$= \frac{100000000}{70710678}\left[1 - \frac{16780316}{10000000000000000}\right.$$

$$- \frac{140789502529928}{(10000000000000000)^2} - \frac{2362492341934991297248}{(10000000000000000)^3}$$

$$- \frac{49554210056561506781339212 96}{(10000000000000000)^4}$$

$$\left.\frac{11641494254312719401178208553241335 04}{(10000000000000000)^5} \quad \&c.\right],$$

$$=1.4142135623730950488016887242096980785696718753769 4 +$$

Let $a = 3$, $n = 2$ and $m = 13$; then

$$\sqrt{3} = \frac{10000000000000}{5773502691896}\left[1 - \frac{8925087775552}{100000000000000000000000000}\right]^{\frac{1}{2}}$$

$$= \frac{10000000000000}{5773502691896}\left[1 - \frac{4462543887776}{100000000000000000000000000}\right.$$

$$- \frac{9957148975163468441113088}{(100000000000000000000000000)^2}$$

$$\left.- \frac{44434214298790798522333481840396812288}{(100000000000000000000000000)^3} - \&c.\right],$$

$$=1.7320508075688772935274463415058723669428052538103 8 +$$

Four additional terms would give the root true to at least one hundred places of decimals.